On the computation of an arch dam by the thin shell theory and finite element method

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THIN SHELL THEORY AND FINITE ELEMENT METHOD***

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&

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RESUME : Dans ce travail, nous montrons comment calculer les contraintes pour un barrage voûte soumis à son poids propre, à une pression hydrostatique et à des charges thermiques. Nous utilisons les trois concepts suivants :

(i) la théorie linéaire de coques de KOITER [1,2] qui permet de formuler le problème dans un domaine plan $\Omega \subset \mathbb{R}^2$. Les inconnues sont les composantes du champ de déplacement des particules de la surface moyenne de la voûte. Le système correspondant d'équations aux dérivées partielles est d'ordre 2 par rapport aux composantes tangentielles et d'ordre 4 par rapport à la composante normale du déplacement.

(ii) des méthodes conformes d'éléments finis. La composante normale du déplacement est approchée dans un espace discret construit à l'aide de triangles d'ARGYRIS ou de HSIEH-CLOUGH-TOCHER complets ou réduits. L'approximation des composantes tangentielles utilise les mêmes espaces, ou, d'autres espaces construits à l'aide de triangles de type (1) ou (2). En outre, nous prenons en compte l'effet de l'intégration numérique.

(iii) l'analyse numérique de problèmes de coques minces de BERNADOU [1, 2]. Ces résultats établissent la convergence et donnent les estimations asymptotiques de l'erreur. En particulier, des conditions suffisantes définissent le degré de précision des schémas d'intégration numérique.

Connaissant le champ des déplacements approchés, nous obtenons une approximation des contraintes en tout point de la voûte.

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ABSTRACT : In this paper, we show how to compute stresses for an arch dam subjected to self-weight, hydrostatic pressure and thermal effects making use of the following three concepts :

(i) the linear thin shell theory of KOITER [1,2] which transform the problem to a plane domain $\Omega \subset \mathbb{R}^2$. Components of displacement field of middle surface of the dam are the unknowns. Corresponding system of Partial Differential Equations is of order 2 with respect to local tangential components, and of order 4 with respect to local normal components of displacement.

(ii) Some conforming finite element methods. The normal component of the displacement is approximated with a discrete space constructed from triangles of ARGYRIS or HSIEH-CLOUGH-TOCHER which are either complete or reduced. The approximation of the tangential components uses the same spaces, or, other spaces constructed from Lagrange triangles of type (1) or (2). Moreover, we take into account the effect of numerical integration.

(iii) the numerical analysis of thin shell problems of BERNADOU [1,2]. These results prove the convergence and give asymptotic error estimates. In particular, sufficient conditions define the degree of accuracy of the numerical quadrature schemes.

From the knowledge of the approximated displacement field, we obtain an approximation of the stresses everywhere in the dam.

1. INTRODUCTION.

Three-dimensional finite element models are currently in use for static analysis of arch dams. Corresponding number of unknowns is great. Therefore, two-dimensional thin shell analysis for non-linear situations or dynamic studies should be more economical.

In this work, we use simultaneously :

- the linear thin shell theory of W.T. KOITER [1,2],
- some conforming finite element methods,
- some results of numerical analysis,

in order to compute a good approximation of the stresses which appear in an arch dam subjected to

- its self weight,
- hydrostatic pressure,
- changes of temperatures.

For clarity, we consider the example of the project of GRAND'MAISON arch dam studied by "COYNE et BELLIER" [1] for "Electricité de France". By analogy, it would be possible to consider other types of arch dams.

In paragraph 2, we recall the definition of the dam as given by "COYNE et BELLIER". This definition appears also in the figure 2.3 and 2.4. With the help of curvilinear coordinates, we define the middle surface and the thickness of the dam.

Next, in paragraph 3, we obtain the expression of the potential energy of external loads using resultants on the middle surface of the dam. Thus, we derive a variational formulation of the continuous problem.

Then, in paragraph 4, we define some associated discrete problems using conforming finite element methods with numerical integration.

In paragraph 5, we indicate how to solve the system and how to obtain approximations of the stresses everywhere in the dam. Particularly, we obtain physical components of the stresses on the upstream and downstream walls of the dam.

Some significant numerical results are given in paragraph 6.

This work is the first step for further studies such as shrinkage, creep, movement of the foundation, earthquake, dynamical behaviour... .

A detailed version of this work is done in BERNADOU-BOISSERIE [3].

2. - GEOMETRICAL DEFINITION OF THE DAM.

In this paragraph, we give the geometrical definition of the arch dam of the project of GRAND'MAISON studied by COYNE et BELLIER [1]. We consider the two following steps :

Step 1 : Definition of the middle surface of the vault.

Let Ω be the open bounded set in a plane $\mathcal{E}^2$ with boundary Γ shown in Figure 2.1. Let $(0,\vec{e}_1,\vec{e}_2,\vec{e}_3)$ be an orthonormal reference system of the usual Euclidean space $\mathcal{E}^3$ as in figure 2.2. Then, the coordinates x^i of any point M of the middle surface S of the dam are given as the components of the mapping

$$\text{(2.1)} \qquad \vec{\phi} : (\xi^1,\xi^2) \in \overline{\Omega} \to \vec{OM} = \vec{\phi}(\xi^1,\xi^2) = x^i(\xi^1,\xi^2)\vec{e}_i$$

by the relations

$$(2.2)\quad \begin{cases} x^1(\xi^1,\xi^2) = \rho_o(\xi^2)\left[e^{\alpha\theta_o|\xi^1|}\cos(\theta_o|\xi^1|+40°) - \cos 40°\right] \\ \qquad + 0{,}269\ z_o\xi^2 - 0{,}0000085\ z_o^3(\xi^2)^3 \\ x^2(\xi^1,\xi^2) = \dfrac{|\xi^1|}{\xi^1}\rho_o(\xi^2)\left[e^{\alpha\theta_o|\xi^1|}\sin(\theta_o|\xi^1|+40°) - \sin 40°\right] \\ x^3(\xi^1,\xi^2) = z_o\xi^2 \end{cases}$$

where

$$(2.3)\quad \begin{cases} \alpha = \mathrm{tg}\ 40°\ ,\ \theta_o = 48°\ 178\ ,\ z_o = 157\ \mathrm{m} \\ \rho_o(\xi^2) = 200 - 0{,}008233(z_o)^2(\xi^2)^2 + 0{,}000029(z_o)^3(\xi^2)^3 \end{cases}$$

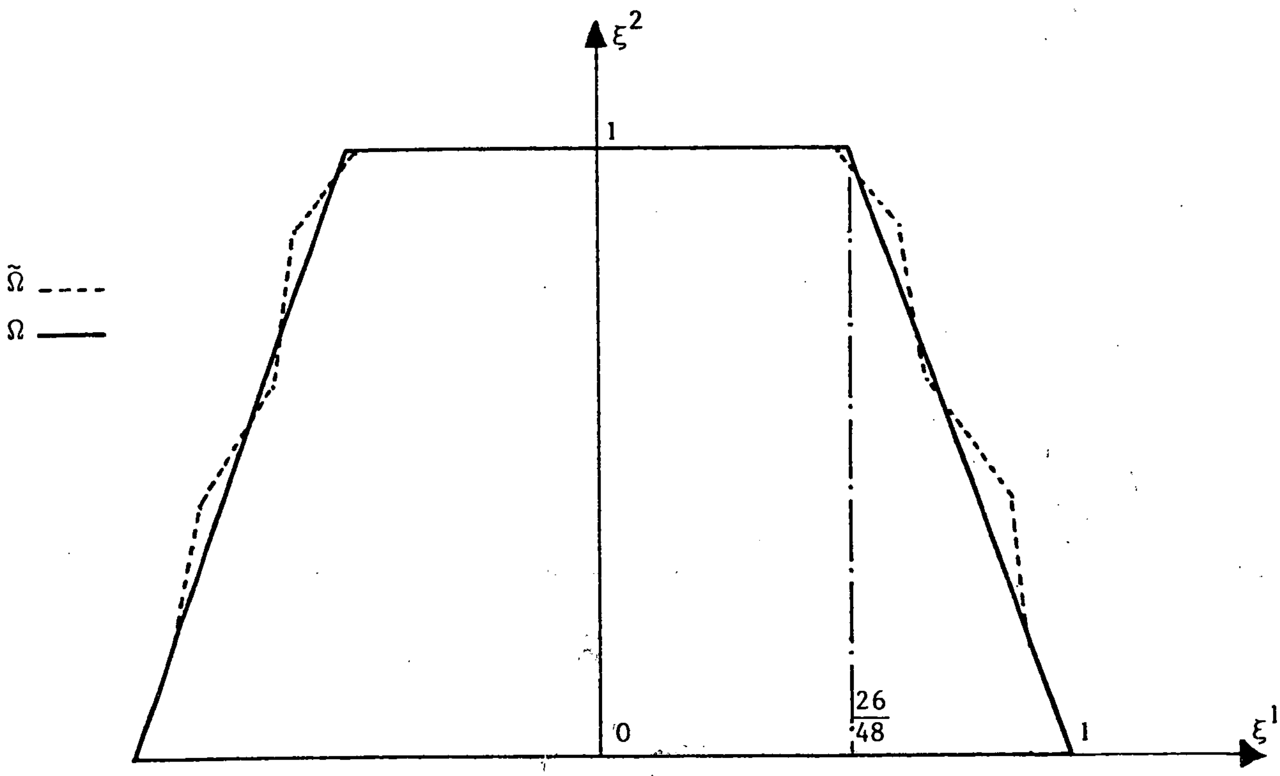


Figure 2.1 : the reference domains Ω and $\tilde{\Omega}$.

The design of COYNE et BELLIER is partly shown in figures 2.3 and 2.4. For simplicity, we have replaced the real unsymmetrical domain $\tilde{\Omega}$ by the symmetrical domain Ω : see figure 2.1.

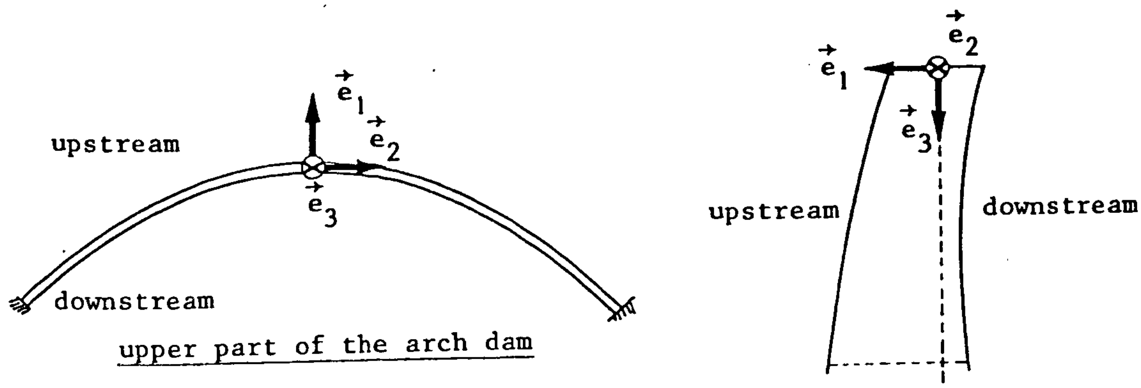


Figure 2.2 : the orthonormal reference system of the space $\mathcal{E}^3$.

Step 2 : Definition of the thickness of the dam.

One can check in the figure 2.4 that the thickness of the dam at the point (ξ^1,ξ^2) is given by

$$
(2.4)\quad \left\{
\begin{aligned}
e(\xi^1,\xi^2) &= 8 + 0{,}248\ z_o\xi^2 - 0{,}000003\ (z_o\xi^2)^3 \\
&\quad + 2.10^{-8}(z_o\xi^2)^2\ [1+ 0{,}003\ z_o\xi^2] \left[\frac{e^{\alpha\theta_o|\xi^1|}-1}{\sin 40^\circ}\ \rho_o(\xi^2)\right]^2
\end{aligned}
\right.
$$

III - VARIATIONAL FORMULATION OF THE PROBLEM.

Since the thickness is small with respect to the other dimensions of the dam, particularly with respect to the curvature radius, we shall use the two-dimensional thin shell theory of KOITER [1,2] to solve the problem. In this theory, the unknowns are the three functions

$$(3.1)\quad u_i : \xi \in \overline{\Omega} \to u_i(\xi) \in \mathbb{R}$$

which represent the covariant components of the displacement $\vec{u} = \vec{u}(\xi)$ of the point $\vec{\phi}(\xi)$, $\xi = (\xi^1,\xi^2)$. More precisely, we introduce the covariant basis of the tangent plane to the middle surface S, i.e.,

$$(3.2)\quad \vec{a}_\alpha = \frac{\partial\vec{\phi}}{\partial\xi^\alpha}\ ,\ \alpha = 1,2,$$

and the corresponding contravariant basis $(\vec{a}^\alpha)$ which is such that

$$(3.3)\quad \vec{a}^\alpha \cdot \vec{a}_\beta = \delta^\alpha_\beta\ ,$$

where δ^α_β is the Kronecker symbol. In the sequel, we shall use Greek letters : $\alpha,\beta,\ldots$, for indices which take their values in the set $\{1,2\}$, while Latin letters : $i,j,\ldots$, will be used for indices which take their values in the set $\{1,2,3\}$. For these indices, we shall use Einstein's convention for summation.

To the previous basis, we associate the unit normal vector to the surface

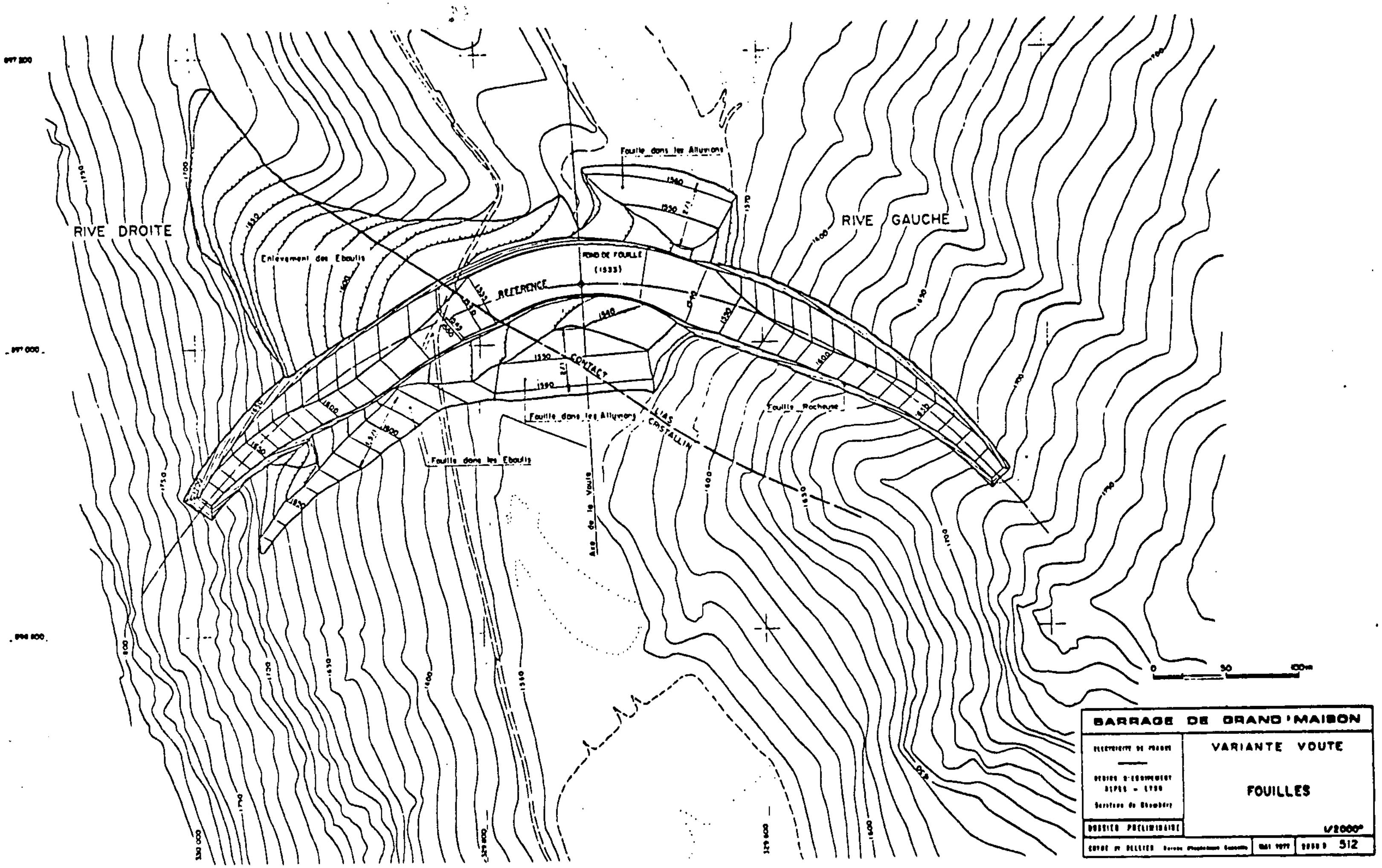


Figure 2.3 - Site plane of the dam (excavations)

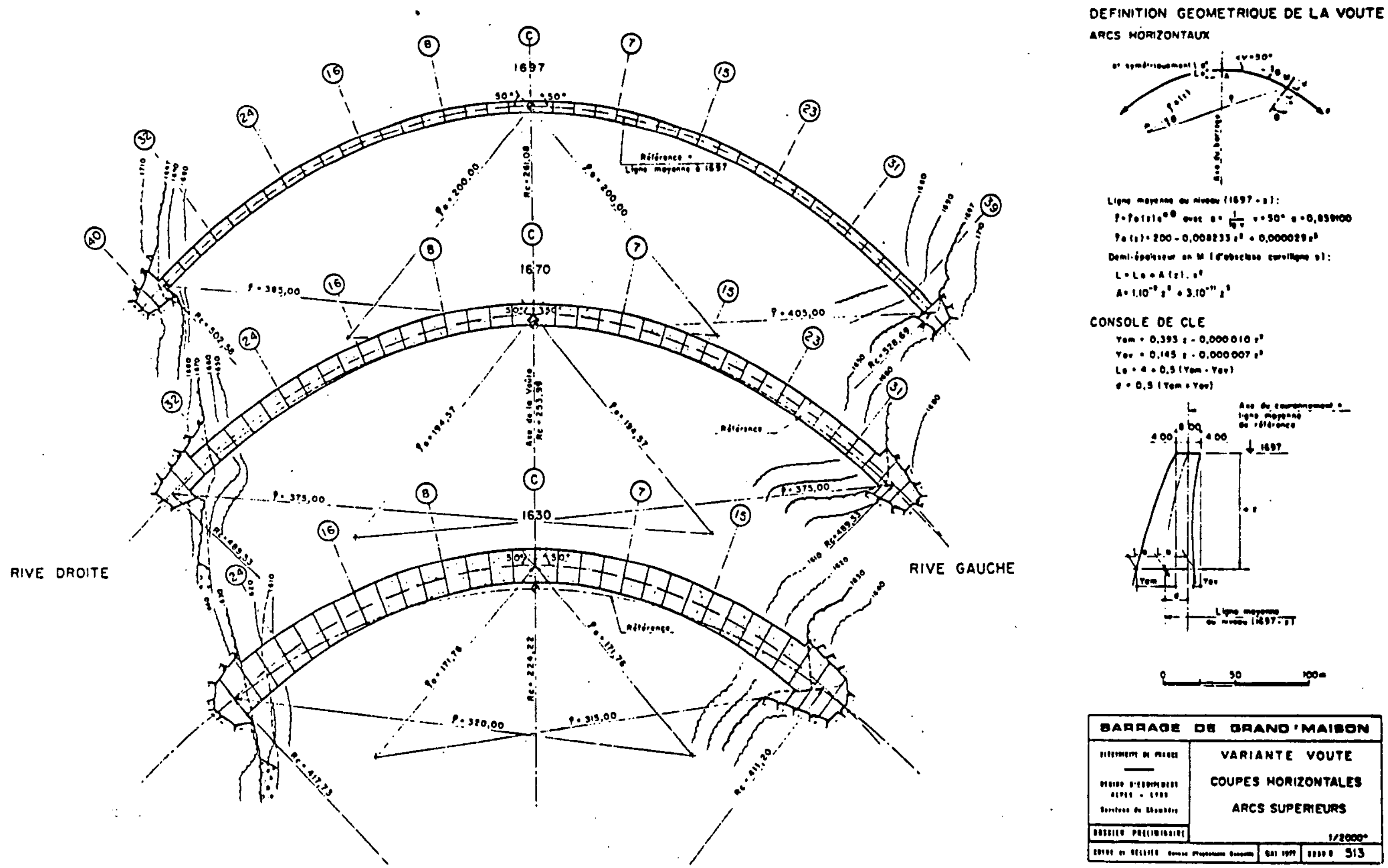


Figure 2.4 - Geometrical definition of the dam (horizontal sections of the upper arcs)

(3.4) $\vec{a}_3 = \vec{a}^3 = \dfrac{\vec{a}_1 \times \vec{a}_2}{\|\vec{a}_1 \times \vec{a}_2\|}$.

Then, the covariant components u_i (cf. (3.1)) are such that

(3.5) $\vec{u} = u_i \vec{a}^i$.

Corresponding to a displacement field $\vec{v}$, the strain energy $\frac{1}{2}\, a(\vec{v},\vec{v})$ of the shell is obtained from

$$(3.6)\quad \left\{\begin{aligned} a(\vec{u},\vec{v}) = \int_\Omega \frac{Ee}{1-\nu^2} \{ & (1-\nu)\gamma^\alpha_\beta(\vec{u})\gamma^\beta_\alpha(\vec{v}) + \nu\gamma^\alpha_\alpha(\vec{u})\gamma^\beta_\beta(\vec{v}) \\ & + \frac{e^2}{12}[(1-\nu)\bar{\rho}^\alpha_\beta(\vec{u})\bar{\rho}^\beta_\alpha(\vec{v}) + \nu\bar{\rho}^\alpha_\alpha(\vec{u})\bar{\rho}^\beta_\beta(\vec{v})]\} \sqrt{a}\, d\xi^1\, d\xi^2, \end{aligned}\right.$$

where e is the thickness of the shell, E is its Young modulus, ν is its Poisson coefficient and $a = a_{11}a_{22}-(a_{12})^2$, $a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta$. The mixed tensors (γ^α_β) and $(\bar{\rho}^\alpha_\beta)$ are obtained from the doubly covariant <u>strain tensor</u> $(\gamma_{\alpha\beta})$ and <u>change of curvature tensor</u> $(\bar{\rho}_{\alpha\beta})$ through the tensorial operations

(3.7) $\gamma^\alpha_\beta = a^{\alpha\nu}\gamma_{\nu\beta}$, $\bar{\rho}^\alpha_\beta = a^{\alpha\nu}\bar{\rho}_{\nu\beta}$, $a^{\alpha\nu} = \vec{a}^\alpha \cdot \vec{a}^\nu$.

The tensors $(\gamma_{\alpha\beta})$ and $(\bar{\rho}_{\alpha\beta})$ are given by

(3.8) $\gamma_{\alpha\beta}(\vec{v}) = \frac{1}{2}\,(v_{\beta|\alpha} + v_{\alpha|\beta}) - b_{\alpha\beta}v_3$,

(3.9) $\bar{\rho}_{\alpha\beta}(\vec{u}) = v_{3|\alpha\beta} - b^\lambda_\alpha b_{\lambda\beta}v_3 + b^\lambda_\alpha|_\beta v_\lambda + b^\lambda_\alpha v_{\lambda|\beta} + b^\lambda_\beta\, v_{\lambda|\alpha}$,

where the second fundamental form $(b_{\alpha\beta})$ is given by

(3.10) $b_{\alpha\beta} = b_{\beta\alpha} = \vec{a}_3 \cdot \dfrac{\partial \vec{a}_\alpha}{\partial \xi^\beta}$,

and where $(|)$ denotes the covariant derivatives with respect to the surface S.

<u>Expression of the work done due to the external loads.</u>

We consider here the effects of thermal loads, water pressure and self weight on the behaviour of the dam. To apply KOITER's theory, we need to obtain an approximation of the work done by these loads through an integration on the thickness. In BERNADOU-BOISSERIE [3], we have derived the following expressions associated to a displacement $\vec{v}$ of the middle surface of the shell :

$$(3.11)\quad \left\{\begin{aligned} f(\vec{v}) = {} & \frac{1-2\nu}{2(1-\nu)}\beta \int_\Omega [eT_1\gamma^\alpha_\alpha + \frac{e^3}{12}\, T_2(2b^\alpha_\eta\gamma^\eta_\alpha - b^\alpha_\alpha\gamma^\eta_\eta - \bar{\rho}^\alpha_\alpha)] \sqrt{a}\, d\xi^1\, d\xi^2 \\ & + \int_\Omega p[\frac{1}{2}\, e_{,\beta}v_\alpha a^{\alpha\beta} - (1-\frac{1}{2}\, eb^\beta_\beta)v_3] \sqrt{a}\, d\xi^1\, d\xi^2 \\ & + \int_\Omega \rho g_o\, e[(a^{12}v_1 + a^{22}v_2)z_o + (\vec{a}_3 . \vec{e}_3)v_3] \sqrt{a}\, d\xi^1\, d\xi^2, \end{aligned}\right.$$

where the three integrals take respectively into account the effects of

(i) the thermal loads : On the middle surface S, we denote T_1 and T_2 the moments of order 0 and 1 through the thickness (in a mathematical sense) of the steady-state temperature distribution ;

(ii) the hydrostatic pressure p. If $\xi^2 = \bar{\xi}^2$ refer to the level of water in the reservoir, then

(3.12) $p=0$ if $0 \le \xi^2 \le \bar{\xi}^2$, $p = \rho_2 g_o z_o (\xi^2 - \bar{\xi}^2)$ if $\bar{\xi}^2 \le \xi^2 \le 1$,

where $\rho_2 = 10^3$ kg/m^3, $g_o = 9,81$ m/s^2, $z_o = 157$ m ;

(iii) the self weight. We take $\rho_1 = 2500$ kg/m^3.

Boundary conditions : for a first approximation, we assume here that the middle surface S is free on the upper part of its boundary and clamped everywhere else. Denoting by Γ_o the clamped part of the boundary Γ, the space of admissible displacement is defined by

$$(3.13) \quad \vec{V} = \{\vec{v} \mid \vec{v} \in (H^1(\Omega))^2 \times H^2(\Omega),\ \vec{v}|_{\Gamma_o} = \vec{0}\ ,\ \frac{\partial v_3}{\partial n}|_{\Gamma_o} = 0\}\ ,$$

where $H^k(\Omega)$ means the Sobolev space

$$(3.14) \quad H^k(\Omega) = \{v \mid D^\alpha v \in L^2(\Omega),\ |\alpha| \le m\}\ .$$

Variational formulation of the continuous problem :

$$(3.15) \begin{cases} \text{For any } T_\alpha \in L^2(\Omega), \text{ find } \vec{u} \in \vec{V} \text{ such that} \\ a(\vec{u},\vec{v}) = f(\vec{v})\ ,\ \forall \vec{v} \in \vec{V}. \end{cases}$$

An extension of BERNADOU-CIARLET [1] gives

Theorem 3.1 : The problem (3.15) has a unique solution.

IV - APPROXIMATION OF THE SOLUTION $\vec{u} \in \vec{V}$ BY CONFORMING FINITE ELEMENT METHODS.

The discrete space $\vec{V}_h$

To any triangulation of the polygonal set $\bar{\Omega}$, we associate a discrete space $\vec{V}_h$ using HSIEH-CLOUGH-TOCHER finite elements, complete or reduced, or ARGYRIS triangle - see CLOUGH-TOCHER [1], ARGYRIS-FRIED-SCHARPF [1] or CIARLET [1, pp. 341, 357, 71]-. These elements are of class $\mathcal{C}^1$, so that the discrete space $\vec{V}_h$ satisfies

$$(4.1) \quad \vec{V}_h \subset \vec{V}.$$

This inclusion involves that the discrete problem

"Find $\vec{u}_h \in \vec{V}_h$ such that $a(\vec{u}_h,\vec{v}_h) = f(v_h)$, $\forall \vec{v}_h \in \vec{V}_h$ "

has one and only one solution.

But, if we consider the expressions (3.6) and (3.11) we can check that is not possible to compute exactly the integrals. So, we use a numerical integration scheme, i.e.,

$$(4.2)\qquad \int_{\Omega} \psi(\xi^1,\xi^2)\,d\xi^1\,d\xi^2 = \sum_{K\in\mathcal{T}_h}\ \sum_{\ell=1}^{L} \omega_{\ell,K}\,\psi(b_{\ell,K})$$

where $\mathcal{T}_h$ is the triangulation of $\bar{\Omega}$, $\omega_{\ell,K}$ are the weights and $b_{\ell,K}$ are the nodes of the scheme.

The discrete problem : Using the numerical integration scheme (4.2) to compute the integrals which appear in (3.6) and (3.11), we define an approximated bilinear form $a_h(\cdot,\cdot)$ and a linear form $f_h(\cdot)$. Then, the new discrete problem is given by

$$(4.3)\qquad \begin{cases} \text{"Find } \vec{u}_h\in\vec{V}_h \text{ such that} \\ a_h(\vec{u}_h,\vec{v}_h) = f_h(\vec{v}_h)\ ,\quad \forall\vec{v}_h\in\vec{V}_h\text{".} \end{cases}$$

Convergence and error estimates

In the figure 4.1, we summarize the convergence and error estimate results which can be obtained by the methods of BERNADOU [1,2]. We denote

$$(4.4)\qquad \|\vec{u}-\vec{u}_h\| = \Big(\sum_{\alpha=1}^{2}\|u_\alpha-u_{\alpha h}\|^2_{1,\Omega}+\|u_3-u_{3h}\|^2_{2,\Omega}\Big)^{1/2}\ ,$$

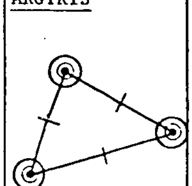
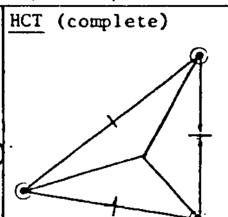
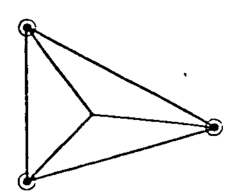
$$(4.5)\qquad E_K(\psi) = \int_K \psi(\xi^1,\xi^2)\,d\xi^1\,d\xi^2 - \sum_{\ell=1}^{L}\omega_{\ell,K}\,\psi(b_{\ell,K})\ ,$$

$$(4.6)\qquad h = \max_{K\in\mathcal{T}_h} h_K,\ h_K = \operatorname{diam}(K)\ ,$$

$$(4.7)\qquad W^{k,q}(K) = \{v\,|\,D^\alpha v\in L^q(K)\ ,\ |\alpha|\le k\}\ .$$

For HCT-elements the numerical integration scheme has to be applied to every subtriangle K_i, i=1,2,3.

Figure 4.1 :
Convergence and error estimates.

	ARGYRIS 	HCT (complete) 	HCT (reduced) 
Error estimate $\|\vec{u}-\vec{u}_h\|$	$O(h^4)$	$O(h^2)$	$O(h)$
Accuracy of the scheme	$\forall\psi\in P_8,\ E_K(\psi)=0$	$\forall\psi\in P_4(K_i),\ E_{K_i}(\phi)=0$	$\forall\psi\in P_4(K_i),\ E_{K_i}(\phi)=0$
Number of nodes of suitable schemes	16 nodes	6 nodes	6 nodes
Regularity	$\vec{u}\|_K\in(H^5)\times H^6$; $T_\alpha\|_K\in W^{4,q}$, $q\ge2$	$\vec{u}\|_K\in(H^3)^2\times H^4$; $T_\alpha\|_K\in W^{2,q}$, $q\ge2$	$\vec{u}\|_K\in(H^2)^2\times H^3$; $T_\alpha\|_K\in W^{1,q}$, $q>2$

V. - IMPLEMENTATION.

The implementation is similar to those described in BERNADOU-BOISSERIE [1,2] and BERNADOU-BOISSERIE-HASSAN [1].

For simplicity, we have assumed that the temperature factors T_1 and T_2 are symmetrical in ξ^1. Then, introducing "pseudo" boundary conditions on $\xi^1 = 0$, we can formulate the problem on the half domain

$$(5.1) \qquad \Omega_1 = \{(\xi^1,\xi^2) \in \Omega \,|\, \xi^1 \geq 0\} \ .$$

The system is solved by a CHOLESKI method using the sky-line band-width factorization (cf. JENNINGS [1]).

From the solution of the system, we obtain an approximation of the displacement, strain tensor and change of curvature tensor at any point of the middle surface S. Then, applying the basic hypothesis of KOITER [1,2], we can derive an approximation of the displacement and mixed stress tensor $\sigma^{*i}_{\ j}$ everywhere in the dam.

Finally, in order to get components having natural physical dimensions, we introduce the so-called right physical components of the stress tensors (cf. TRUESDELL [1])

$$(5.2) \qquad \sigma^*(ij) = \sqrt{\frac{g_{ii}}{g_{jj}}}\ \sigma^{*i}_{\ j} \quad \text{(no summation on i or j)},$$

where

$$(5.3) \qquad \begin{cases} \sqrt{g_{\alpha\alpha}} = \sqrt{a_{\alpha\alpha}}\ (1 - \dfrac{b_{\alpha\alpha}}{a_{\alpha\alpha}}\ \xi^3) & \text{(no summation on } \alpha) \\ \sqrt{g_{33}} = 1 \end{cases}$$

and where ξ^3 means the thickness coordinate.

VI. NUMERICAL EXPERIMENTS.

In this section, we present some results obtained by using ARGYRIS' triangle to approximate the three components of the displacement. Other numerical experiments using reduced-H.C.T. triangle are in progress.

Figures 6.1a to 6.1h show the distribution of stresses on the upstream and downstream faces of the arch dam subjected to the hydrostatic pressure. The level of water in the reservoir is assumed to be 152 meters so that $\overline{\xi}^2 = 0.032$ (see (3.12)) meanwhile

$$E = 2.10^6 \text{ ton/m}^2 \ , \ \nu = 0.2.$$

The figures 6.1a to 6.1h are associated to triangulations with 8, 18, 32, 50 triangles : the corresponding results - see table 6.1 - show the excellent approximation properties of the ARGYRIS element. Particularly, the results obtained from a coarse triangulation with only 8 triangles are really acceptable and non expensive (about F.F. 115).

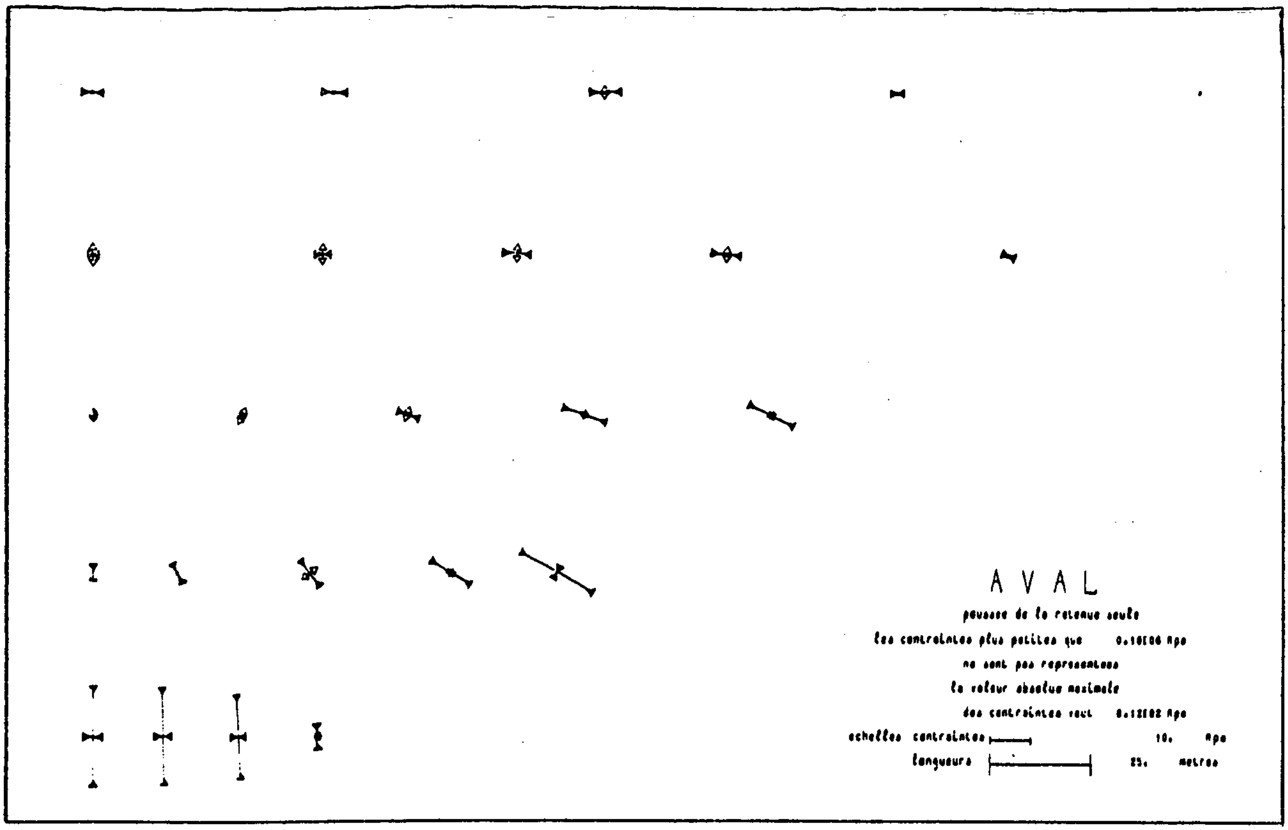


Fig. 6.1a : Stress distribution on downstream face (8 triangles, ν =0.2)

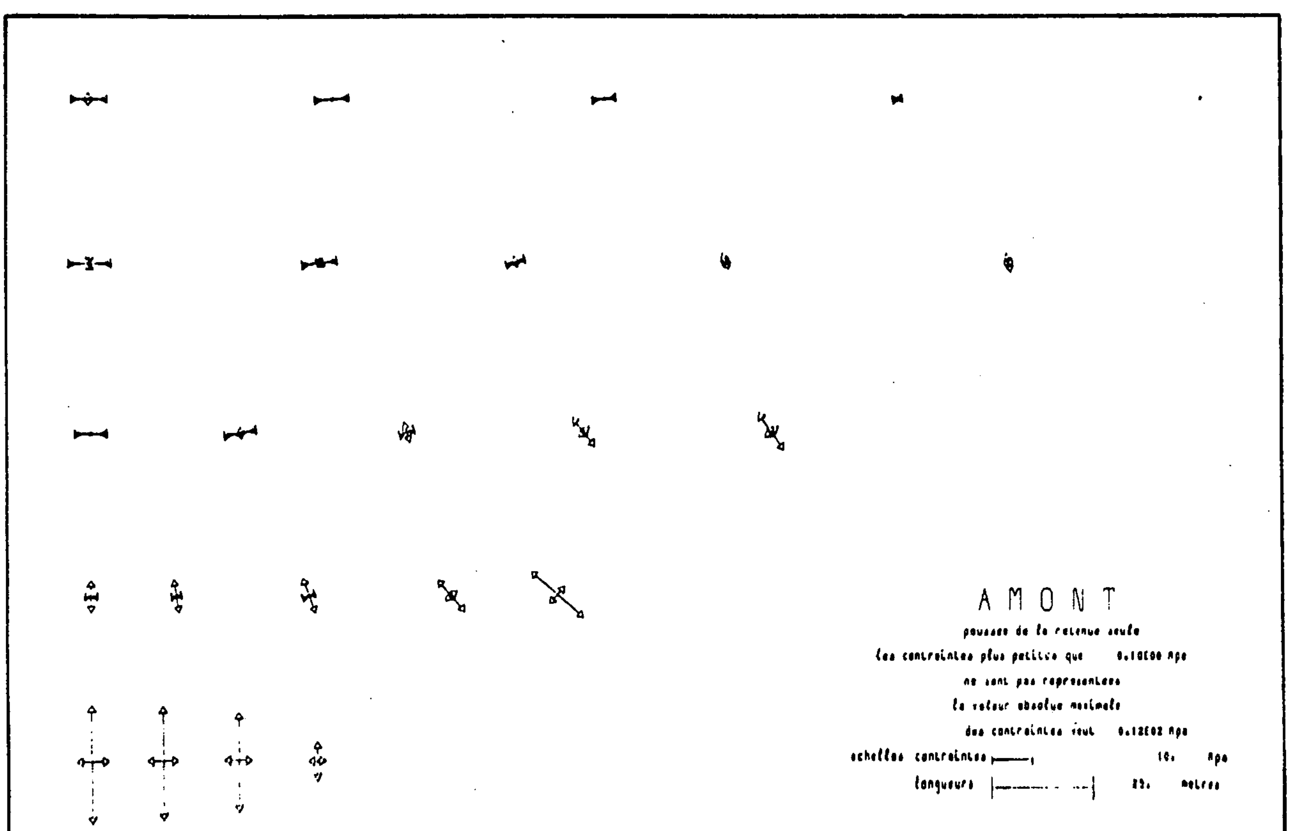


Fig. 6.1b : Stress distribution on upstream face (8 triangles, ν = 0.2)

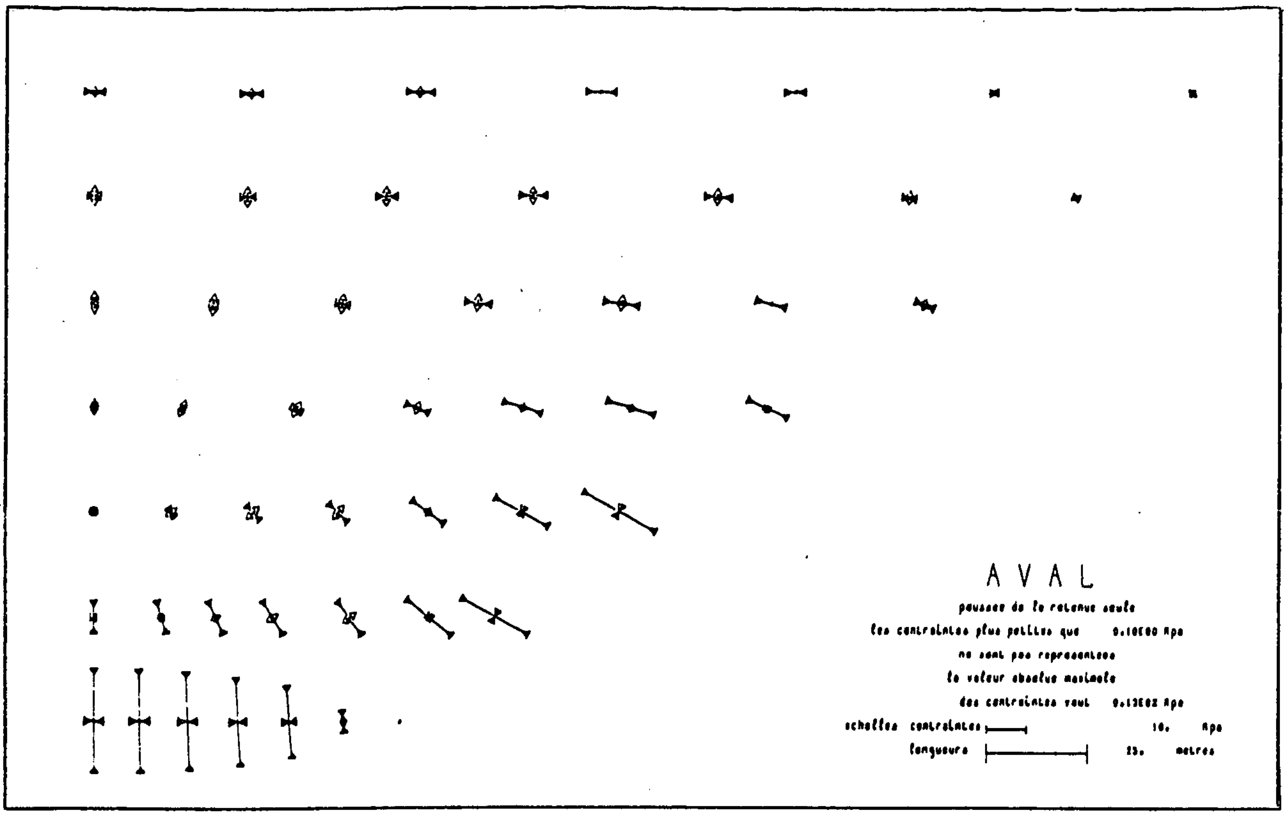


Fig. 6.1c : Stress distribution on downstream face (18 triangles, $\nu = 0.2$)

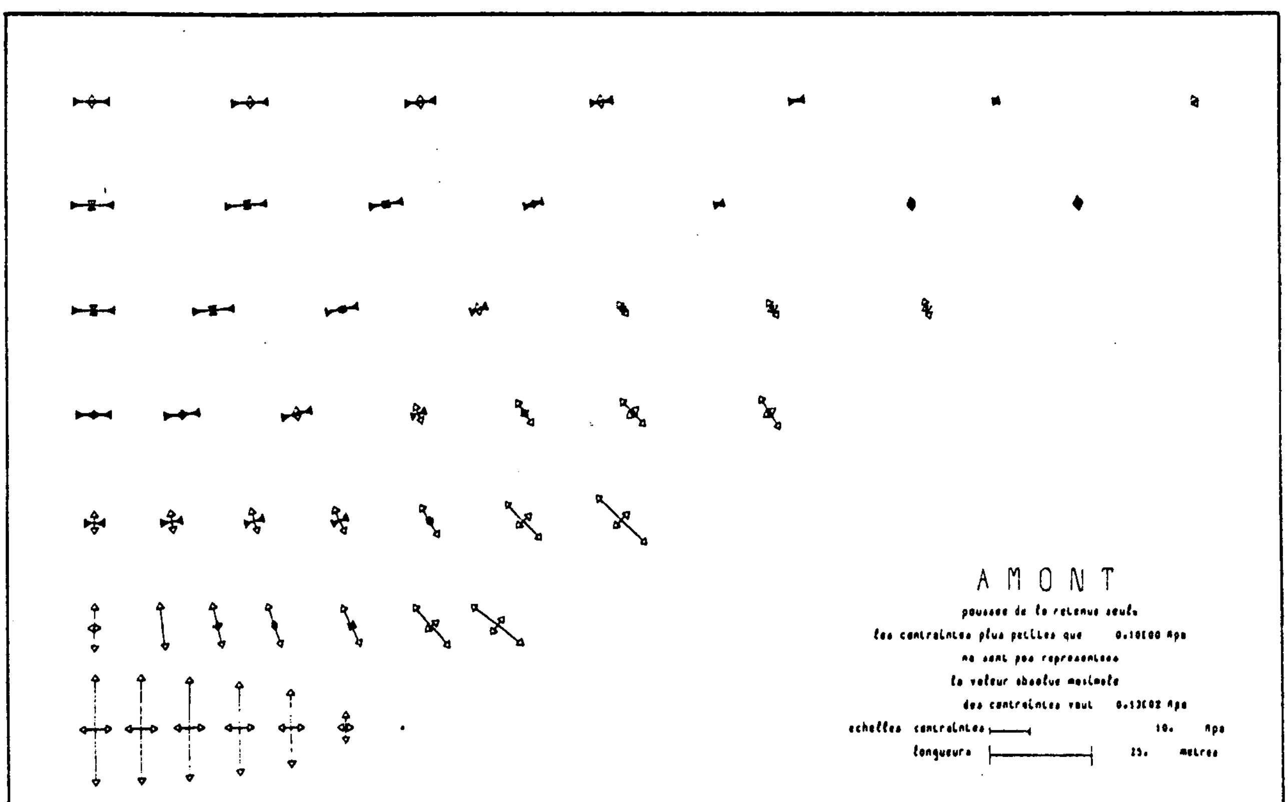


Fig. 6.1d : Stress distribution on upstream face (18 triangles, $\nu = 0.2$)

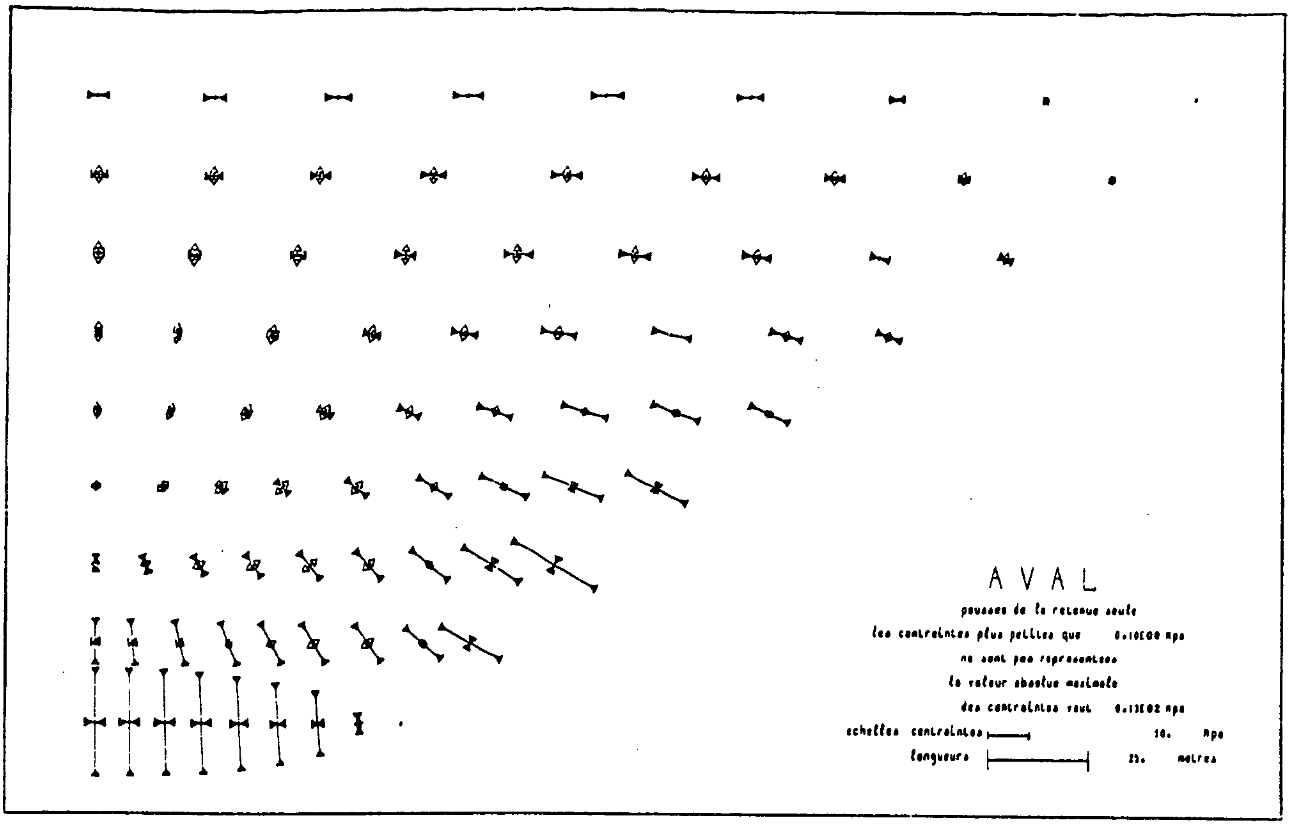


Fig. 6.1e : Stress distribution on downstream face (32 triangles, $\nu = 0.2$)

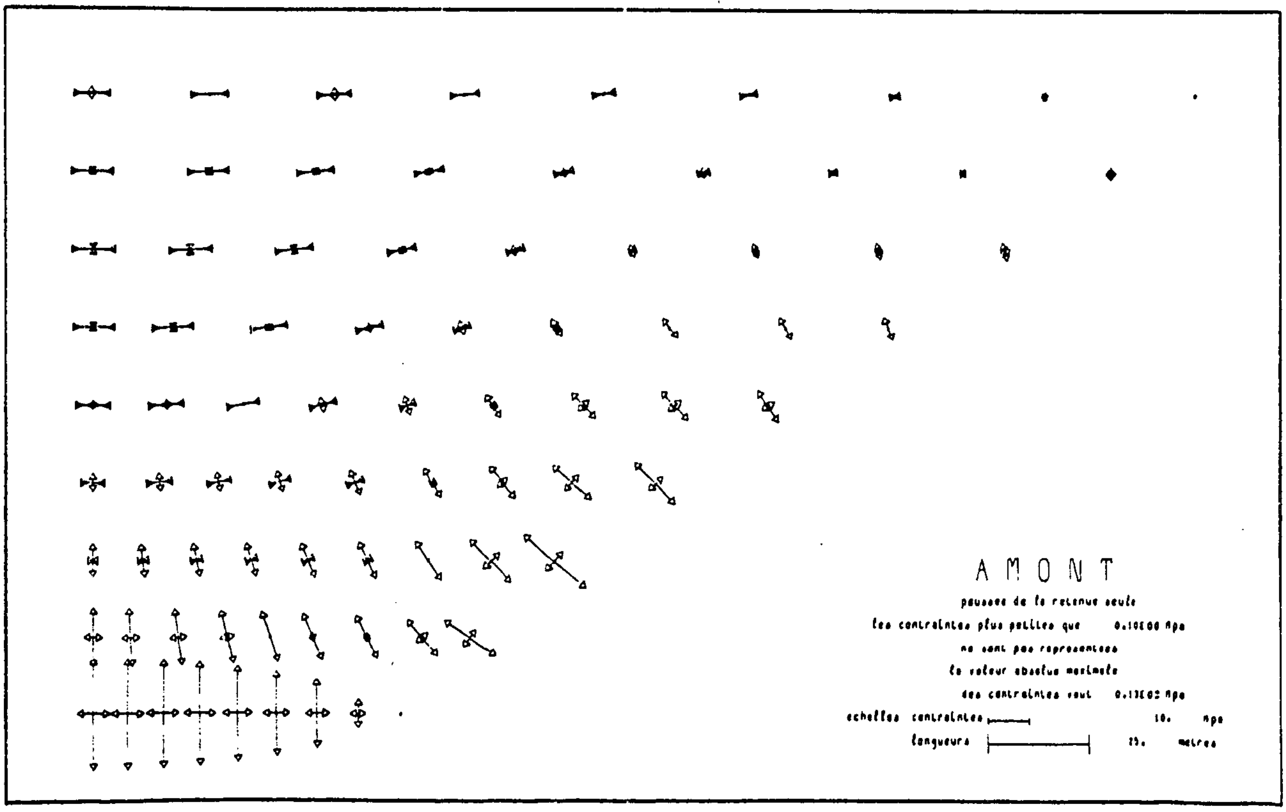


Fig. 6.1f : Stress distribution on upstream face (32 triangles, $\nu = 0.2$)

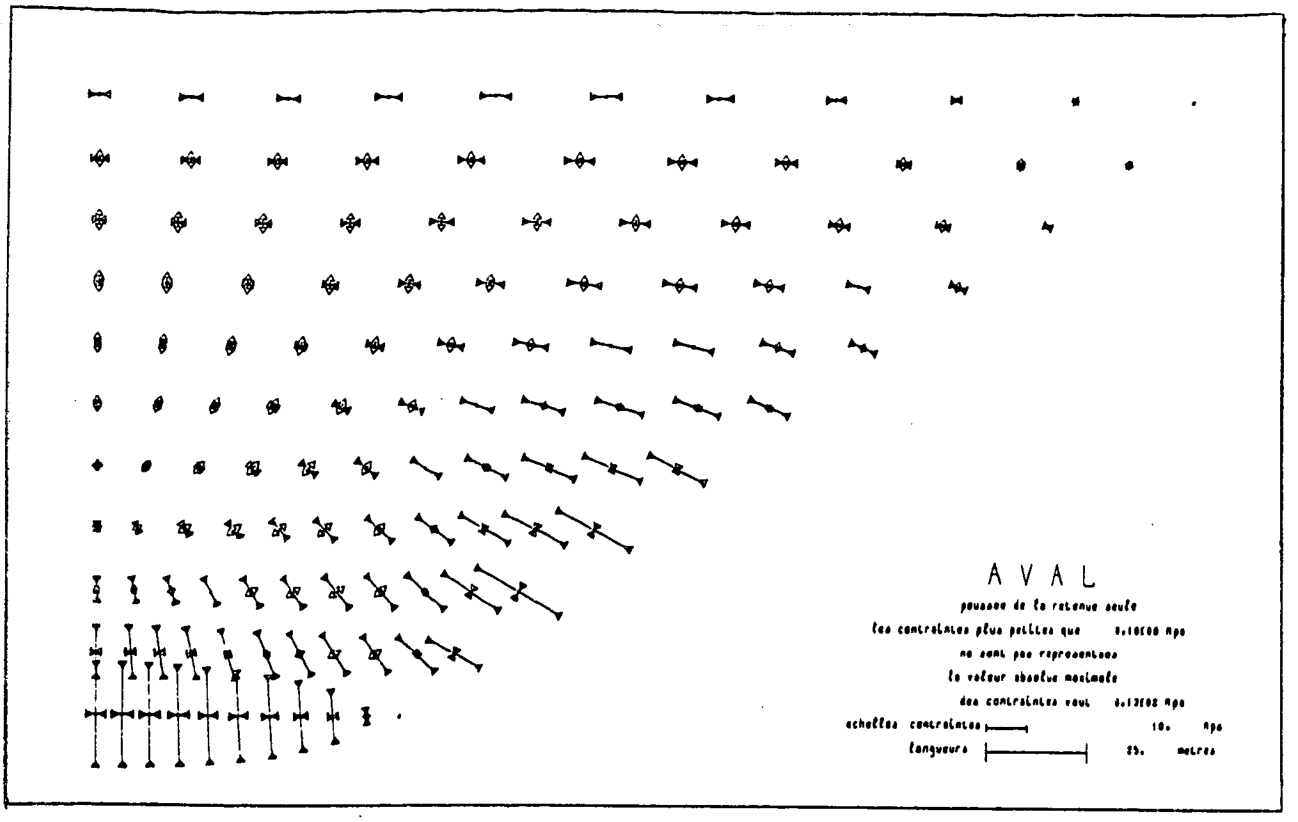


Fig. 6.1g : Stress distribution on downstream face (50 triangles, $\nu = 0.2$)

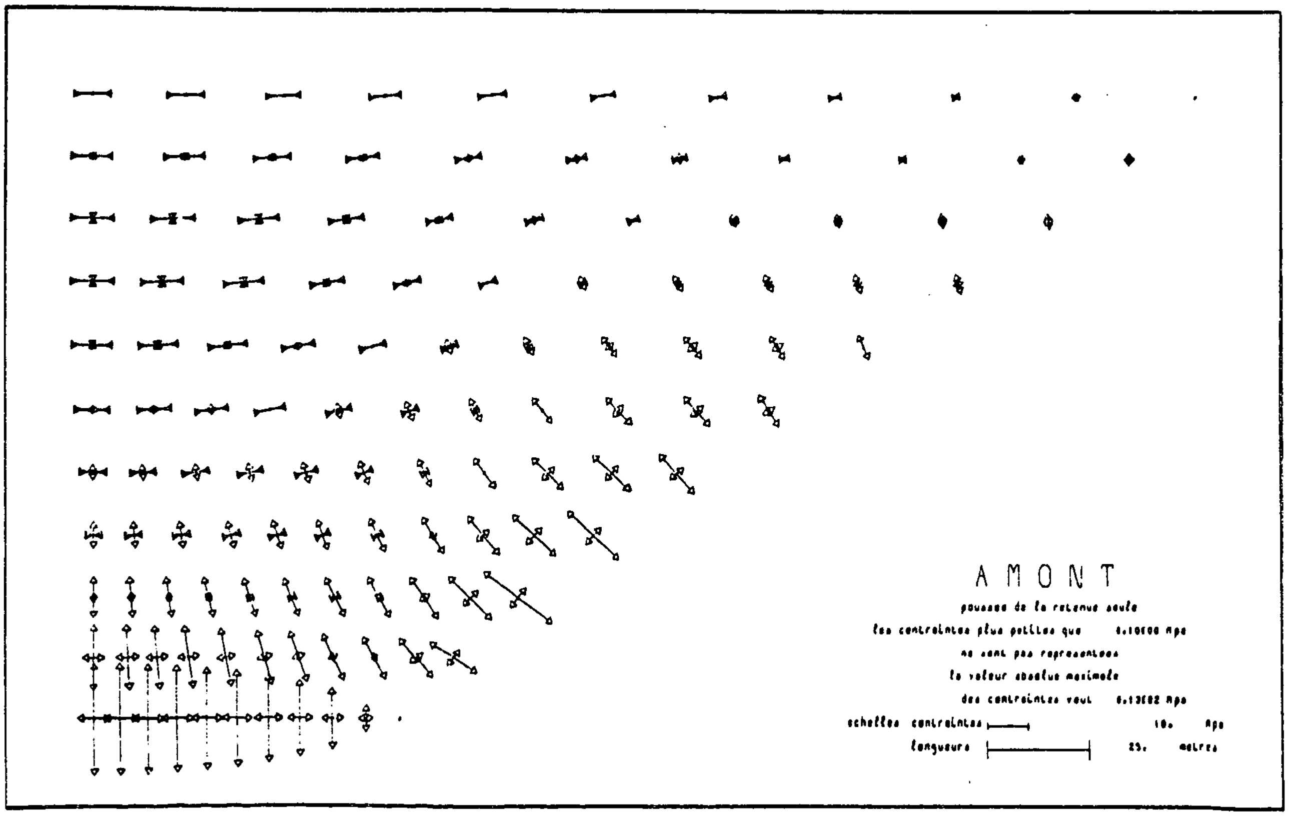


Fig. 6.1h : Stress distribution on upstream face (50 triangles, $\nu = 0.2$)

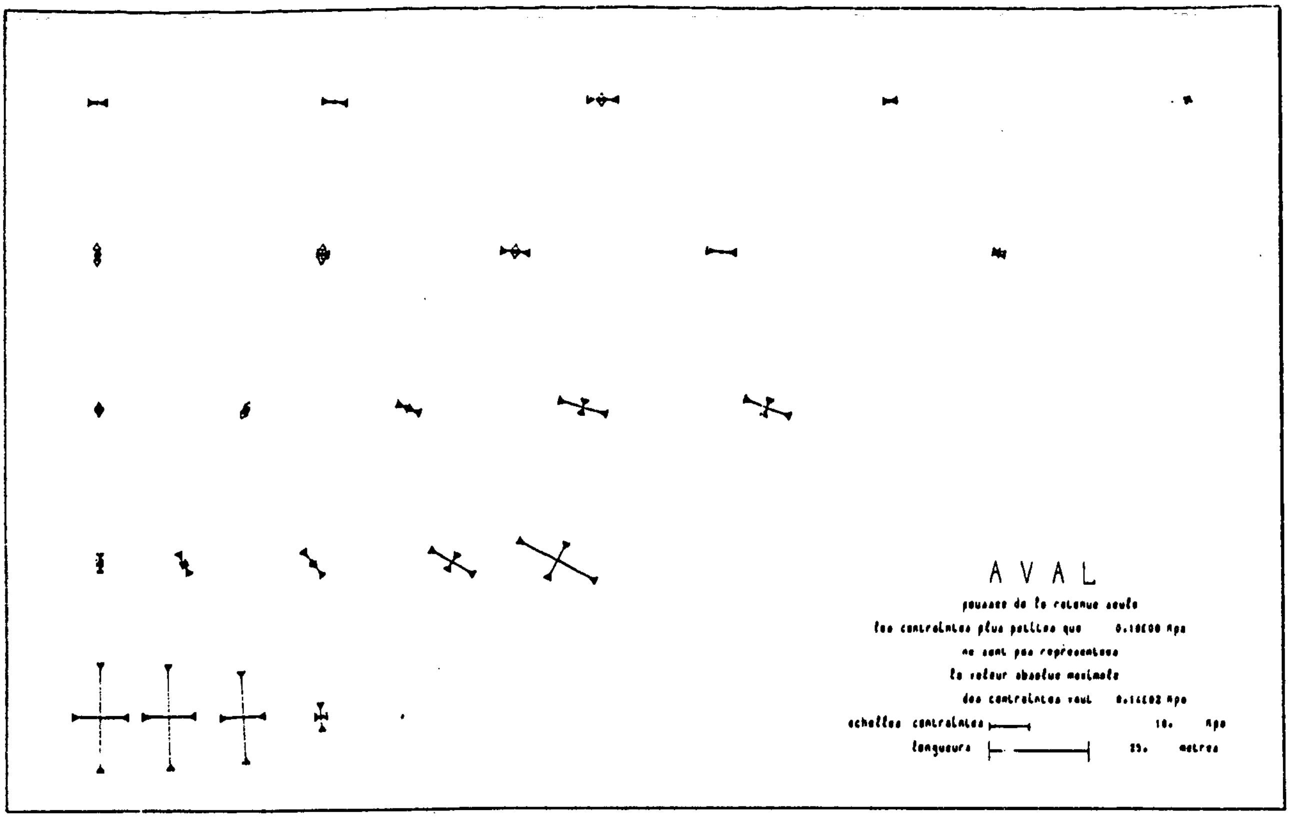


Fig. 6.2a : Stress distribution on downstream face (8 triangles, $\nu = 0.495$)

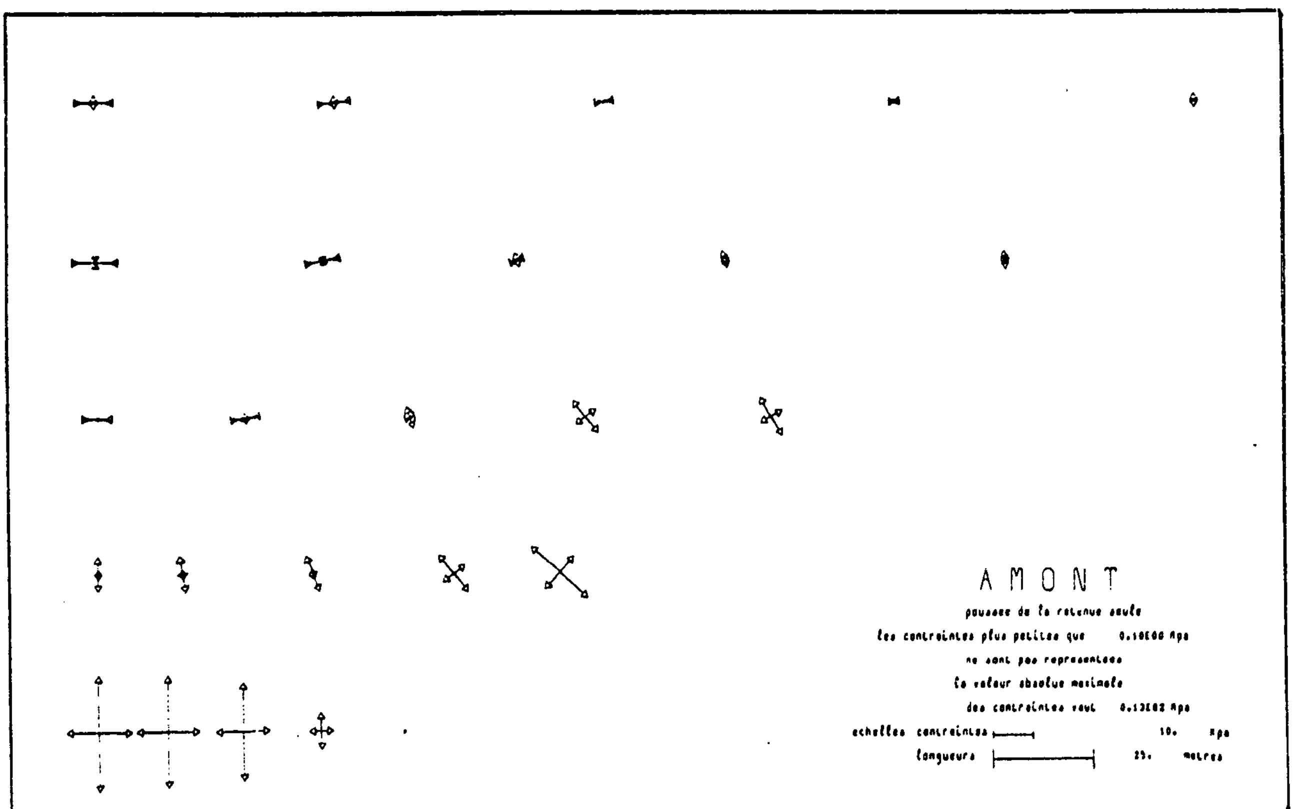


Fig. 6.2b : Stress distribution on upstream face (8 triangles, $\nu = 0.495$)

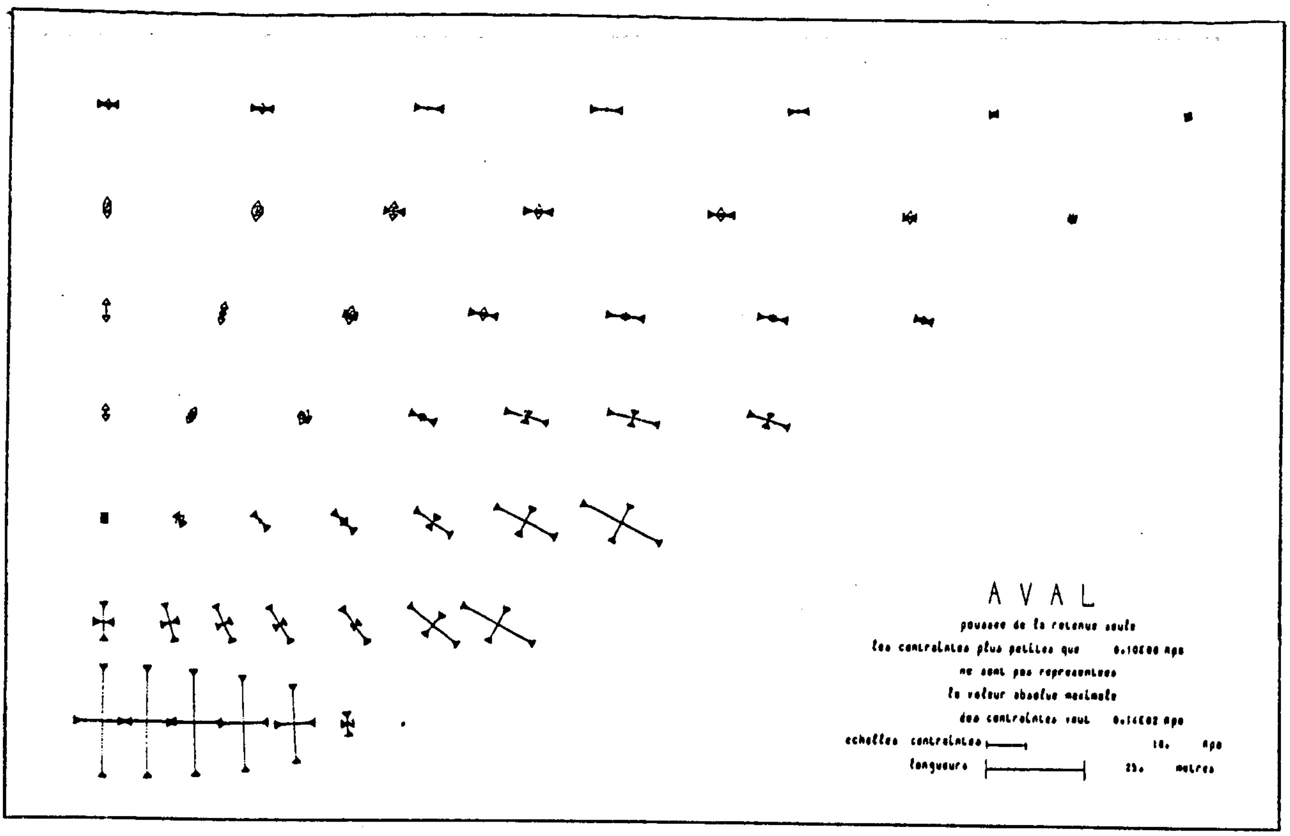


Fig. 6.2c : Stress distribution on downstream face (18 triangles, ν =0.495)

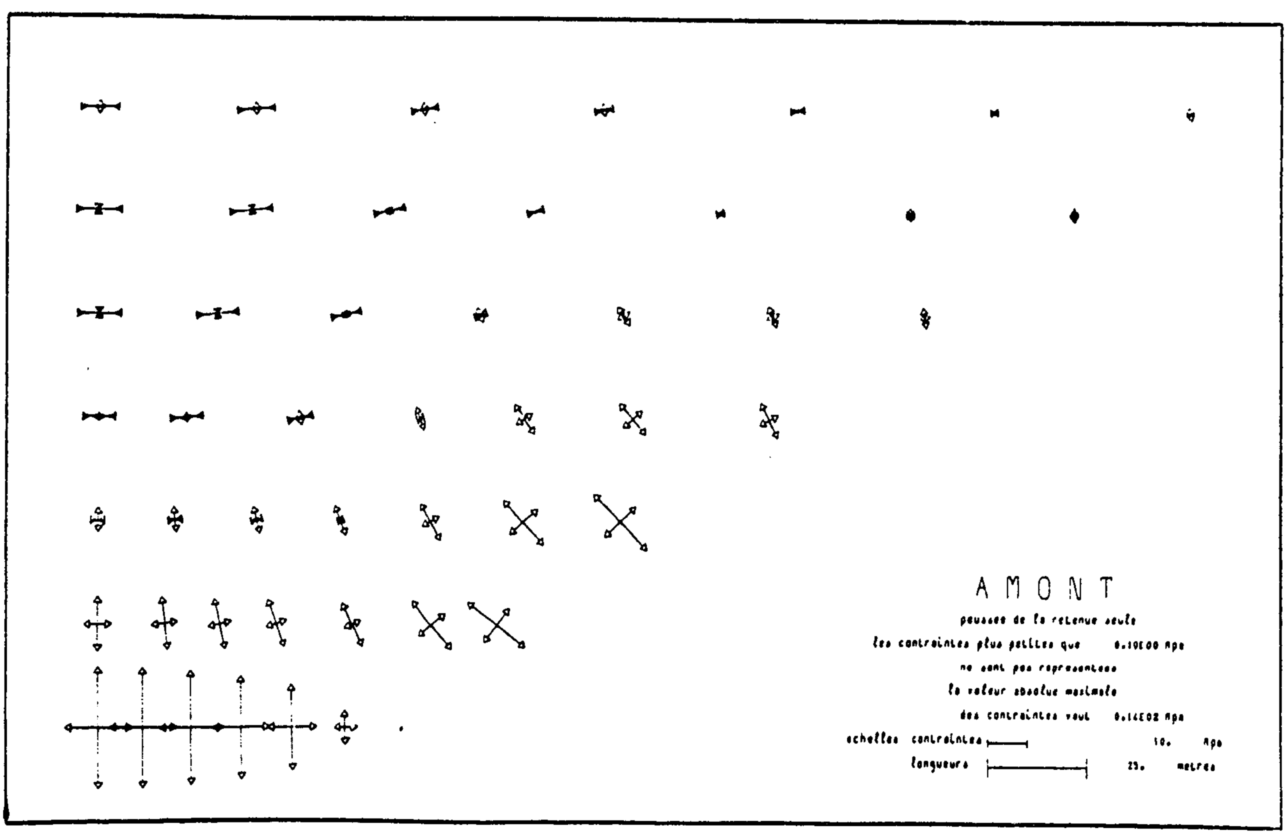


Fig. 6.2d : Stress distribution on upstream face (18 triangles, ν = 0.495)

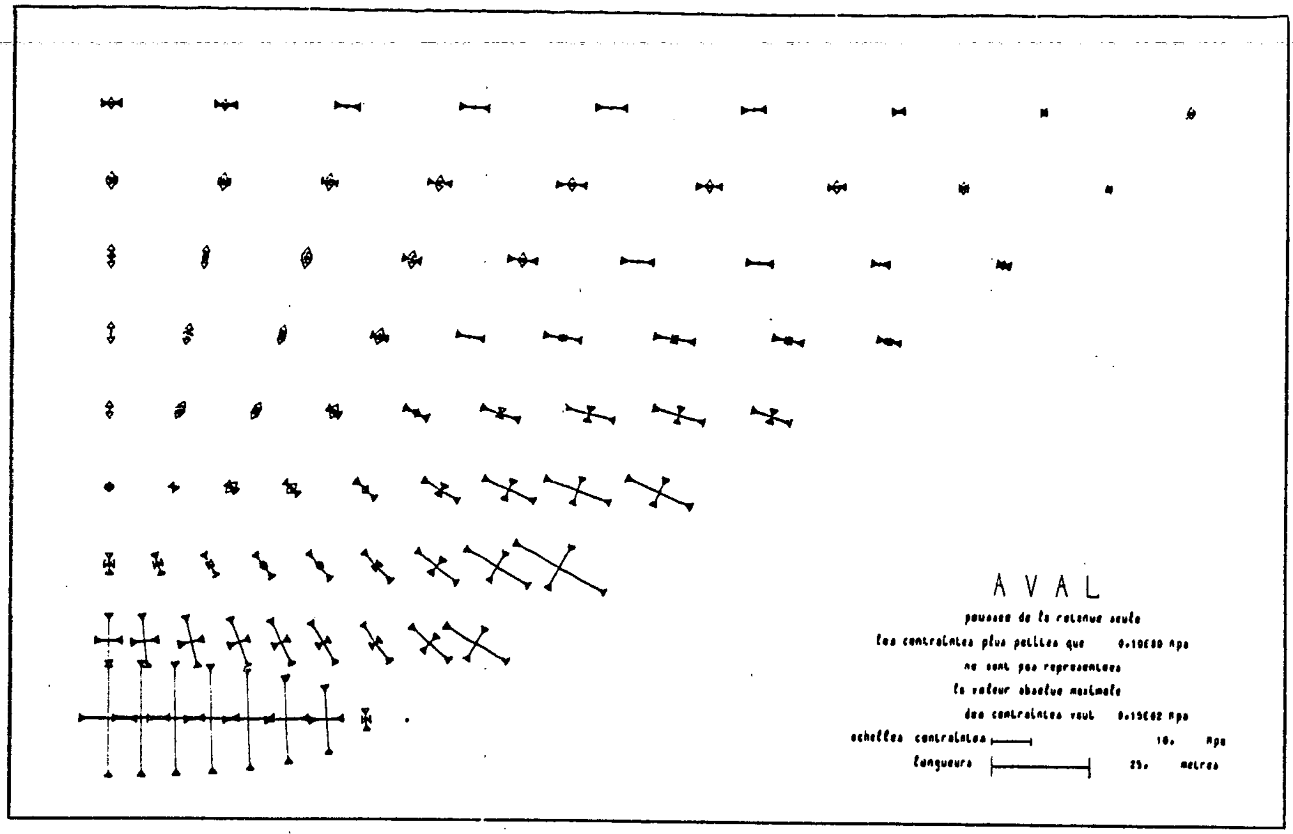


Fig. 6.2e : Stress distribution on downstream face (32 triangles, $\nu = 0.495$)

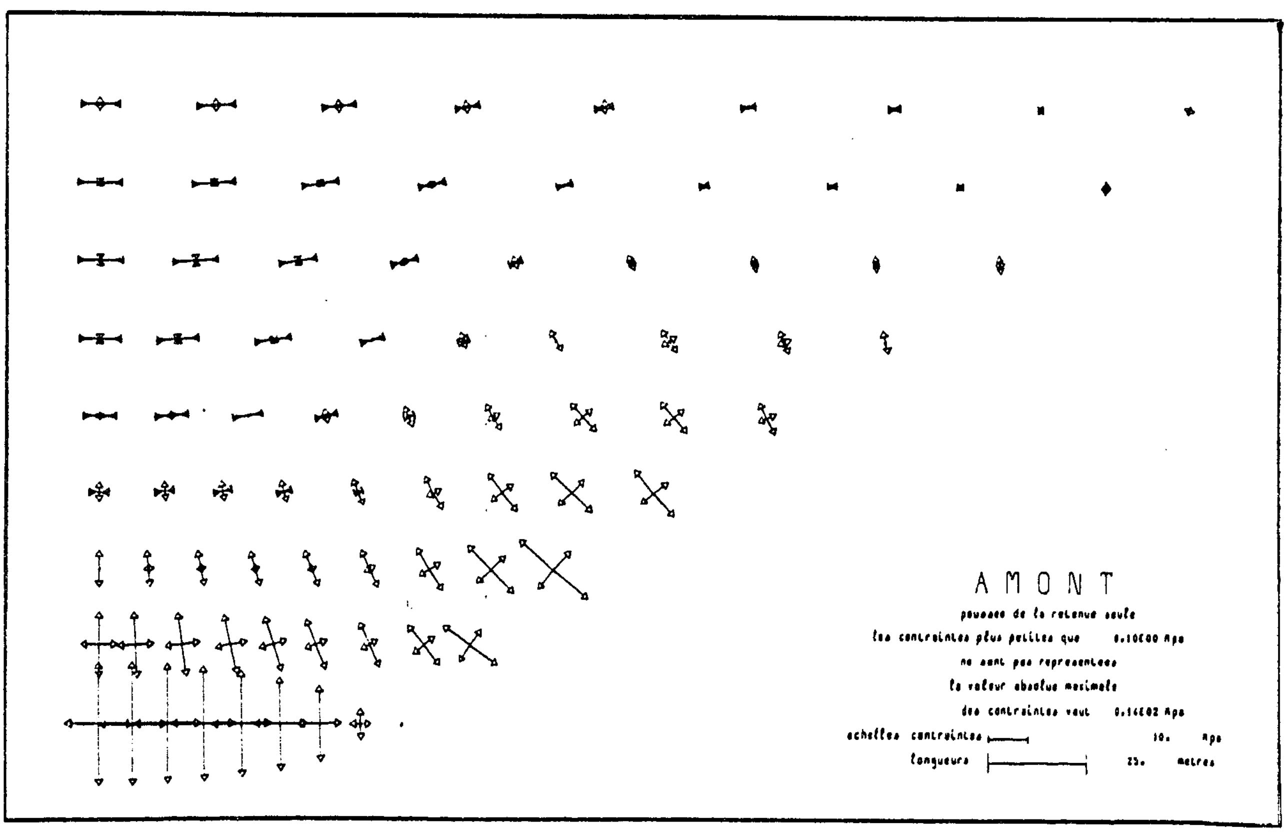


Fig. 6.2f : Stress distribution on upstream face (32 triangles, $\nu = 0.495$)

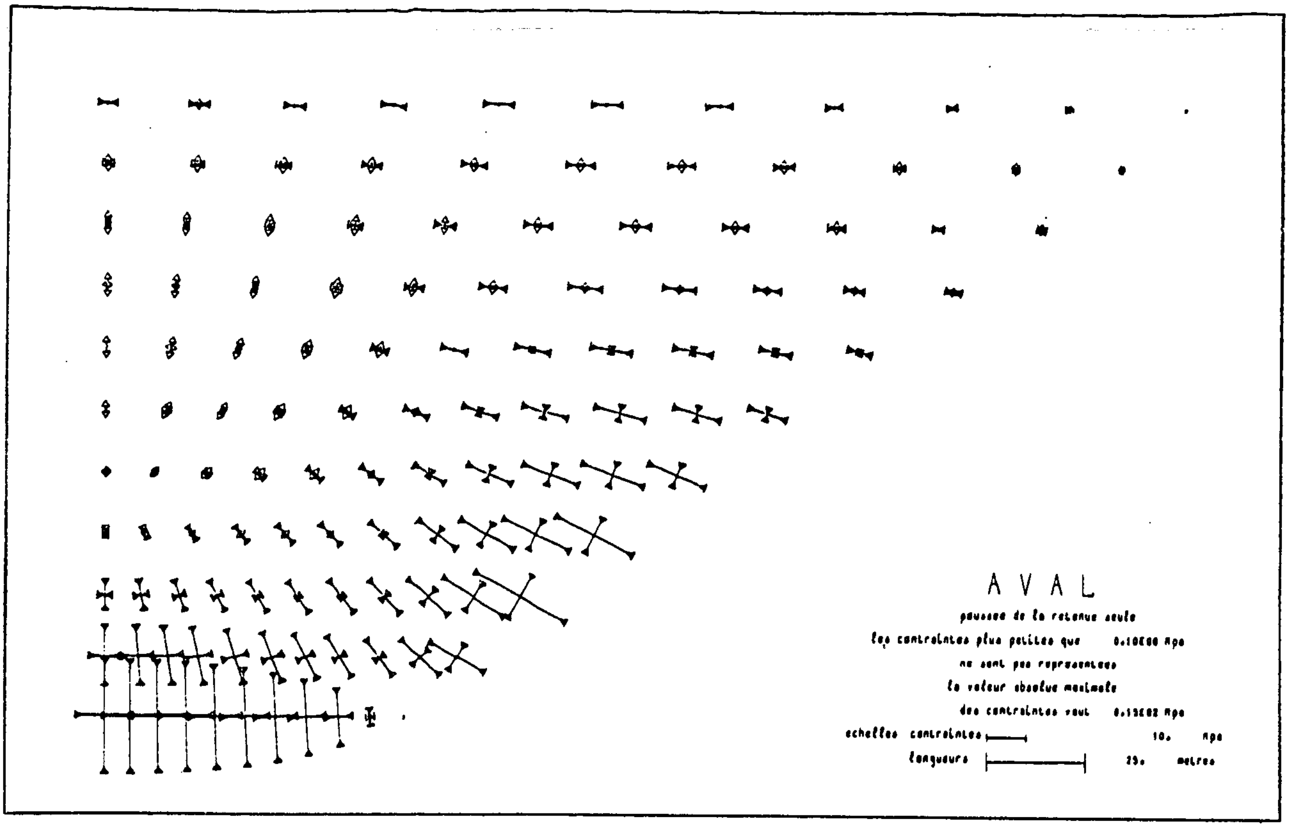


Fig. 6.2g : Stress distribution on downstream face (50 triangles, $\nu = 0.495$)

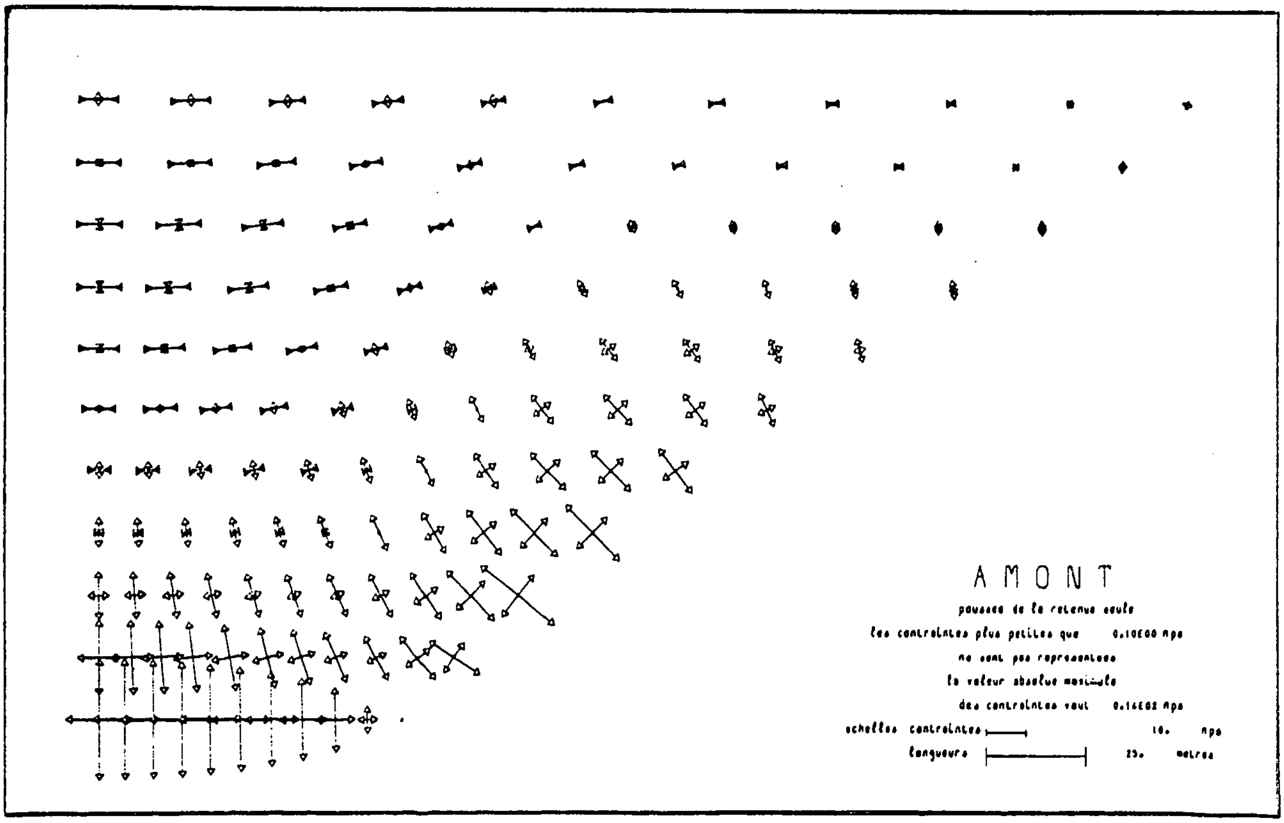


Fig. 6.2h : Stress distribution on upstream face (50 triangles, $\nu = 0.495$)

Let us observe that these numerical experiments can be qualitatively compared with the experimental results given in RYDZEWSKI [1, p. 639] for a similar arch dam.

Other kinds of loadings such that thermal or weight effects will be considered in BERNADOU-BOISSERIE [3].

In addition table 1 gives results for $\nu = 0.495$ instead of $\nu = 0.2$, the other data being unchanged. This test has been done to answer a question which was asked for a long time among the civil engineering profession : what is the behaviour of an arch dam when Poisson's coefficient is increasing closed to 0.5 ? Although the concrete Poisson's coefficient is closed to 0.2, this question was interesting since many experimental models (for instance, at E.D.F.) were built using rubber. Corresponding stress distributions are indicated in Figures 6.2a to 6.2h.

Number of triangles	Top of crown cantilever-Arch Stresses (MN/m^2) Upstream	Downstream	Base of crown cantilever-Arch Stresses (MN/m^2) Upstream	Downstream	Computing time (on IBM D033)
8(ν=0.2) 8(ν=0.495)	-4.46 -4.78	-2.81 -2.56	+2.35 +6.43	-2.43 -6.86	0mn 17
18(ν=0.2) 18(ν=0.495)	-4.60 -4.91	-2.77 -2.61	+2.50 +6.80	-2.59 -7.23	0mn 41
32(ν=0.2) 32(ν=0.495)	-4.61 -4.96	-2.80 -2.62	+2.53 +6.86	-2.62 -7.29	1mn 20
50(ν=0.2) 50(ν=0.495)	-4.62 -4.98	-2.81 -2.63	+2.55 +6.90	-2.64 -7.34	2mn 20

Table 6.1 : Effect of triangulation refinements (ν=0.2 or ν=0.495)

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